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FINESTRUCTURE OF TRANSIENT WAVES IN A RANDOM MEDIUM: THE CORRELATION AND SPECTRAL DENSITY FUNCTIONS

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INTRODUCTION

This is essentially a progress report on a theoretical investigation of the propagation of transient waves in a random medium. The emphasis in this study is on applications to sonic-boom propagation, particularly as regards the effect of atmospheric turbulence on the sonic-boom waveform. The analysis is general, however, and is applicable to other types of waves besides sonic-boom waves.

The phenomenon of primary concern in this investigation is the finestructure of the wave. The first figure shows what is meant by the finestructure.

1. PRELIMINARIES

The uppermost sketch shows a typical transient waveform as it might look if observed under ideal conditions; i.e., in the absence of scattering, refraction, or any other mechanisms which might act to distort the waveform. This is what I call the incident wave.

When a wave propagates in a real medium, such as the atmosphere, a random high-frequency structure, called the finestructure, generally apears on the waveform, giving it the appearance shown in the bottom sketch. The finestructure is defined formally, at least insofar as the present investigation is concerned, as the difference between the observed wave and its ensemble average, and generally has an appearance much like that shown in the middle sketch.

The finestructure is believed to arise as a result of scattering and redistribution of wave energy by the random inhomogeneities of the medium. The finestructure is particularly important in connection with sonic-boom propagation, since the strong high-frequency component that often appears near the front of the N-wave may contribute appreciably to the perceived noisiness of the sonic boom.

The finestructure is most conveniently characterized by its statistical properties, such as the variance, the standard deviation, the correlation function, and the spectral density function. The spectral density function is the quantity that is most relevant to the question of annoyance, particularly as regards sonic-boom waves, since it contains information on the frequency content of the finestructure. The spectral density function is the quantity of primary interest in this study.

Incident Wave
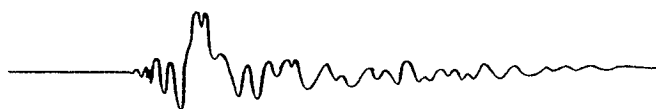
Finestructure
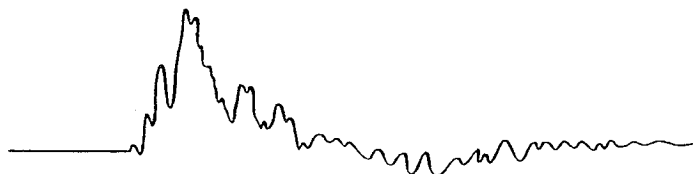
Observed Wave

2. FORMULATION

The investigation is based on a relatively simple mathematical model. In this model the medium is assumed to be one-dimensional, with properties that vary only with the spatial coordinate; i.e., the medium is assumed to be independent of time. The medium is also assumed to be quiescent, which means that only thermal scattering is being considered. (Mechanical turbulence, which gives rise to inertial scattering, is ignored.) In addition, the medium is assumed to be non-dissipative. Non-linear effects are disregarded.

The starting point of the analysis is the time-dependent scalar wave equation in one spatial dimension x. (Note that derivatives are denoted by superscripts rather than by subscripts.) Here w is the wave function; f is the source term.

The sound speed c_*, which is assumed to be a random function of x, is written as shown, where c is a constant reference speed and μ is a stationary random function of x having zero mean and unit variance. (The angle brackets denote an ensemble average.) The paramter ϵ is the standard deviation of the index of refraction of the medium, and is assumed to be small.

It should be pointed out that, in the atmosphere, the standard deviation of the acoustic index of refraction is typically about one part in one thousand, so that the assumption that ϵ is small, which is to say, the assumption that the medium is only weakly inhomogeneous, is appropriate for the study of sonic-boom propagation.

The initial conditions reflect the assumption that the medium is at rest prior to the initial disturbance.

$$c_*^{-2} w^{tt} - w^{xx} = f; \ t > 0, \ -\infty < x < +\infty$$

$$c_*(x) = \frac{c}{1 - \epsilon\mu(x)}, \ <\mu> = 0, \ <\mu^2> = 1$$

$$w(x,0) = w^t(x,0) = 0$$

3. PERTURBATION METHOD

The assumption of a weakly inhomogeneous medium allows the problem to be solved by a perturbation method. To begin, assume that the wave function w has an expansion (which may be asymptotic) in powers of ϵ. By substituting this expansion back into the wave equation, equations for the coefficients w_0, w_1, etc. are derived in the usual way. The first two such equations, for w_0 and w_1, are shown. It turns out to be sufficient to carry out this procedure only as far as the first-order term (the w_1 term). The wave function w is then approximated by the first two terms of the expansion, as indicated on the last line.

Instead of specifying the source term f, it turns out to be more convenient to specify the zeroth-order wave function; i.e., the w_0 term, which corresponds to the incident wave. It is written as shown, where h, which is an arbitrary function, defines the waveform of the incident wave.

Writing the incident wave in this form is equivalent to specifying a source term that is concentrated at the origin, having a time dependence determined by the function h.

With w_0 specified in terms of the function h, the right-hand side of the equation for w_1 is determined, provided that μ is regarded as a known function. That equation can be solved by the method of characteristics. That method is well known, and so will not be described here. Instead, let's look next at how an expression for the correlation function of the finestructure is derived, once an expression for w_1 has been obtained.

$$w(x, t; \epsilon) = w_0(x, t) + \epsilon w_1(x, t) + \epsilon^2 w_2(x, t) + \ldots$$

$$c^{-2} w_0^{tt} - w_0^{xx} = f$$

$$c^{-2} w_1^{tt} - w_1^{xx} = 2c^{-2} \mu w_0^{tt}$$

$$\vdots$$

$$w_0(x, t) = h(t - |x|/c)$$

$$w = w_0 + \epsilon w_1 + \mathcal{O}(\epsilon^2)$$

4. CORRELATION FUNCTION

To get a general expression for the correlation function of the finestructure, start with the expression for w and average it, noting that the average of w_1 is zero. (The average of w_1 is zero because it is linear in the random function μ, which has average zero.) Note that the average field differs from the incident field by terms of order ϵ^2.

Next, subtract to get an expression for $\tilde{w}$, the fluctuating field, defined as the difference of w and its average. The quantity $\tilde{w}$ corresponds to the finestructure. An expression for the temporal correlation function of the finestructure at any point x is then obtained by forming, at x, the product of $\tilde{w}$ with itself at the two time values t_1 and t_2.

The averaging procedure used here is not the usual ensemble averaging. Instead, it's a travel-time-corrected averaging procedure that I call asynchronous ensemble averaging. The essential idea of the method is that, instead of measuring time with respect to some universal reference time, such as the time that the wave is emitted by the source, it is measured, for each wave in the ensemble, relative to the time that that wave arrives at the observer. The advantage of using a travel-time-corrected averaging procedure of this type is that it avoids certain spurious effects that arise as a consequence of averaging over an ensemble of waves that have become dispersed due to variations in travel time among the different members of the ensemble.

A travel-time corrected averaging procedure similar to the one used here was used by Allan Pierce in one of his papers on sonic-boom propagation.

I should mention also that asynchronous averaging has the effect of renormalizing the perturbation series for the wave function, which results in the elimination of some secular terms from the expression for the first-order field.

In order to apply asynchronous averaging in deriving an expression for the correlation function of the finestructure, one replaces t_1 by $\tau + r$, where τ is the travel time from the source (the origin) to the observation point x, and r is the time relative to the arrival time at which the observation is being made. The variable t_2 is replaced by $\tau + r + s$, where s denotes the separation time between the two observation times. The result is an expression for the correlation function of the finestructure, which is denoted by K. Note that terms of order ϵ^3 have been dropped from the expression for K.

The spectral density function is then obtained by taking the cosine transform of K with respect to s.

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$$w = w_0 + \epsilon w_1 + \mathcal{O}(\epsilon^2)$$

$$<w> = w_0 + \mathcal{O}(\epsilon^2)$$

$$w - <w> \equiv \widetilde{w} = \epsilon w_1 + \mathcal{O}(\epsilon^2)$$

$$<\widetilde{w}(x,t_1)\widetilde{w}(x,t_2)> = \epsilon^2 <w_1(x,t_1)w_1(x,t_2)> + \mathcal{O}(\epsilon^3)$$

$$t_1 = \tau + r; \quad t_2 = \tau + r + s; \quad r \geq 0, s \geq 0$$

$$K(x,r,s) = <\widetilde{w}(x,\tau+r)\widetilde{w}(x,\tau+r+s)>$$

$$= \epsilon^2 <w_1(x,\tau+r)w_1(x,\tau+r+s)>$$

5. GENERAL RESULTS

The result of the analysis sketched above is the general expression shown for the correlation function of the finestructure. The function γ that appears in this formula is a temporal correlation function that's related to the spatial correlation function of the medium by the change of variable indicated. The function h determines the waveform of the incident wave, as discussed previously; the primes denote derivatives. The lower limit of the second inner integral; i.e., $\bar{\xi}$, is defined as shown.

This formula is valid provided that the propagation path length x is much greater than the integral scale, or outer scale, L, of the medium. (This condition is satisfied in virtually all cases of practical interest.) Note also that this expression is valid only for r positive. When r is negative, i.e., before the wave has arrived, the correlation function is zero.

This formula, which expresses the correlation function of the finestructure in terms of the incident waveform and the correlation function of the medium (the last two quantities both being arbitrary), is the main result of this investigation.

The spectral density function of the finestructure is obtained by taking the cosine transform of K with respect to s (the separation variable).

One point about this expression that's worth noting is that it is independent of the propagation range x. The reason for this is that, according to the theory, by the time the incident wave has propagated a distance of the order of an outer scale length into the medium the finestructure has become, in a statistical sense, fully developed. Beyond this range, i.e., for x >> L, there is, for all practical purposes, no further evolution of the statistical properties of the finestructure.

The correlation function K is, however, strongly dependent on the magnitude of the randomness of the medium, as is shown by the presence of the term ϵ^2 on the right-hand side of this expression.

Note also that the correlation function K depends on r (the elapsed time since onset), as well as on s, the separation time. The spectral density function of the finestructure will therefore also depend on r. The correlation and spectral density functions of the finestructure are thus time-dependent functions. This is a consequence of the fact that the finestructure of a transient signal is generally a non-stationary random process.

The formula shown expresses the function K as the sum of two iterated integrals. Of these, the two inner integrals involve only the function h. If that function has a sufficiently simple form, then the inner integrals can be evaluated analytically, after which only a pair of single integrals is left to deal with.

One case in which this type of simplification is feasible is

when the incident wave has the form of a simple ramp function,
since then the function h is piecewise constant. The relevant
calculations have been carried out, and the results are shown on
the next figure. (This type of simplification is also feasible
when the incident wave has the form of an N-wave, but the
calculations for that case have not been done.)

GENERAL RESULTS

$$K(r,s) = \frac{1}{2}\epsilon^2 \left[\int_0^r \gamma(\xi) \int_\xi^r h'(\eta - \xi) h'(\eta + s) d\eta d\xi \right.$$
$$\left. + \int_{-s}^r \gamma(\xi + s) \int_{\bar{\xi}}^r h'(\eta - \xi) h'(\eta) d\eta d\xi \right]$$

$$\gamma(\xi) = < \phi(s)\phi(s + \xi) >, \ \phi(s) = \mu\left(\frac{1}{2}cs\right)$$
$$\bar{\xi} = \max(0, \xi), \ x >> L, \ r \geq 0, \ s \geq 0$$

6. RAMP FUNCTION

The sketch shows the function h(t) for the case of a ramp-function incident wave. The amplitude P and the rise time δ are shown on the sketch, as are the two time values r and r+s at which the finestructure is evaluated in order to form the two-point correlation function. The results described here are for the case in which the points r and r+s are both behind the rise phase of the incident wave, as indicated. This is the case of most practical importance.

As has been mentioned, the two inner integrals in the expression for K shown on the previous vu-graph can, in this case, be evaluated exactly. The result of that calculation is the first equation shown. Note that, although K is written generally as a function of both r and s, the right-hand side of this expression is in fact independent of r, showing that the finestructure is a stationary stochastic process in this case.

An expression for the spectral density function of the finestructure, denoted by D, is obtained by taking the cosine transform of K with respect to s. The result is given by the second equation. The function β is the transform of γ.

Of these two quantities, the spectral density function; i.e., D, is, as has already been emphasized, the quantity of primary interest in this study. The expression for this quantity will be examined more closely in a moment. First, however, let's take a quick look at what happens to these two formulas in the limit as δ goes to zero, i.e., as the ramp function becomes a step function.
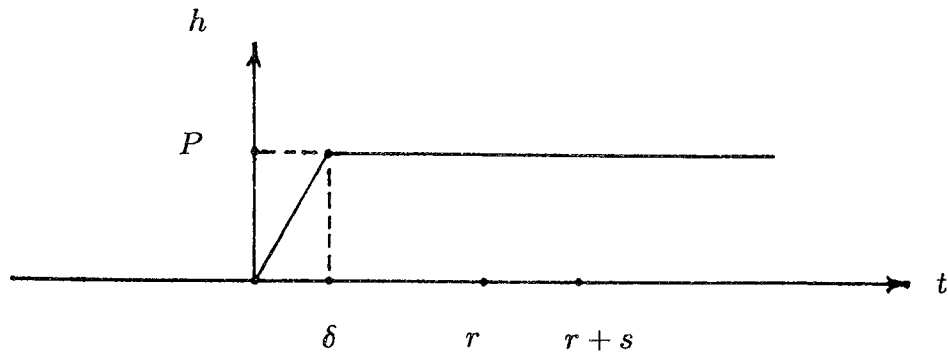
The limiting forms of K and D, denoted by $K_0(s)$ and $D_0(\omega)$, respectively, are easy to calculate, and are given by the last two equations. The function $K_0(s)$ is then the correlation function, while $D_0(\omega)$ is the spectral density function, of the finestructure for the case in which the incident wave is a step function.

It should be pointed out that no analytical problems arise in calculating these limits. There is thus no problem with singular behavior of the finestructure in the limit as the ramp-function incident wave becomes a step function.

These last two formulas show that, when the incident wave is a step function, the correlation function of the finestructure is proportional to the correlation function of the medium, and, similarly, the spectral density function of the finestructure is proportional to the spectral density function of the medium.

When the incident wave has the form of a ramp function, however, with a non-zero rise time, these simple relations no longer obtain. This situation, particularly as regards the spectral density function, will be examined in more detail in the next section.

$$K(r,s) = \frac{\epsilon^2 P^2}{2\delta^2} \int_0^\delta (\delta - \xi)\left[\gamma(s - \xi) + \gamma(s + \xi)\right] d\xi$$

$$D(r,\omega) = \epsilon^2 P^2 \; \frac{1 - \cos(\delta\omega)}{(\delta\omega)^2}\, \beta(\omega)$$

$\delta \to 0:$

$$K(r,s) \to \frac{1}{2}\, \epsilon^2 P^2\, \delta(s) \equiv K_0(s)$$

$$D(r,\omega) \to \frac{1}{2}\, \epsilon^2 P^2\, \beta(\omega) \equiv D_0(\omega)$$

7. RAMP FUNCTION (cont'd)

In order to examine some of the implications of the result obtained above for the ramp-function finestructure spectrum D, it's convenient to write the expression for that quantity in the form shown on the first line. In this formula the function D_0 is the finestructure spectrum for the case of zero rise time, i.e., for the step-function incident wave, as described above. The function A determines how the finestructure spectrum differs from D_0 when the rise time δ is non-zero.

The function A is sketched in the figure. Note that it is effectively zero when its argument is greater than about 2π. What this means is that (referring to the formula for D) the finestructure spectrum is effectively zero at all frequencies for which $\delta\omega$ is greater than about 2π, or, what is equivalent, when f, the frequency in Hz, is greater than about $1/\delta$.

This result shows that, the greater the rise time of the incident wave, the less high-frequency energy there is in the finestructure. A couple of examples of this effect, involving parameter values typical of sonic-boom propagation, are shown: When the rise time is one millisecond, there is effectively no energy in the finestructure spectrum at frequencies above about 1 kHz. If the rise time is increased to two milliseconds, then the lower limit of the frequency range for which the finestructure spectrum is devoid of energy drops to 500 Hz.

Note, however, that, since the function A is equal to unity when its argument is equal to zero, the low-frequency portion of the finestructure spectrum is relatively unaffected by changes in the rise time.

Since it's generally the high-frequency portion of the acoustic spectrum that's most annoying to the human ear, what these results imply is that, at least for a ramp-function wave in one dimension, increasing the rise time of the incident wave will tend to reduce the annoyance associated with the finestructure. Whether this result holds in three dimensions as well remains to be seen, but if it does it has obvious implications as regards the idea of shaping the sonic boom in order to reduce its annoyance.

The results just described could, of course, be expressed just as well in terms of time scales, rather than in terms of frequency. Expressed in those terms, the results imply that a ramp-function wave propagating in a random medium will contain very little structure having time scales smaller than the rise time. This prediction agrees with observations of sonic-boom and other types of transient waves, which rarely show any appreciable structure having time scales smaller than the rise time.

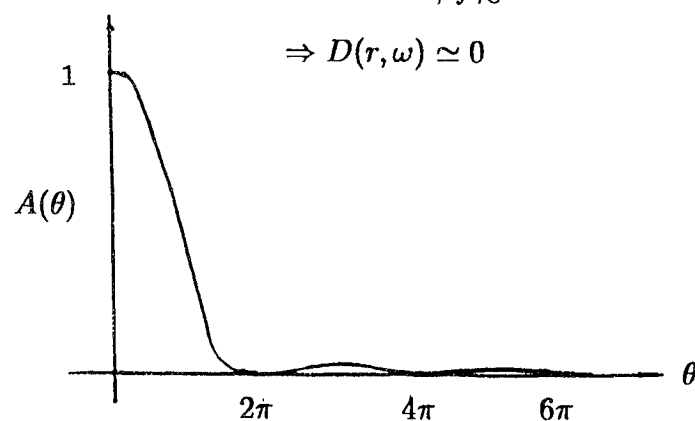$$D(r,\omega) = A(\delta\omega)\, D_0(\omega)$$

$$A(\theta) = 2\ \frac{1 - \cos\theta}{\theta^2}$$

The function $A(\delta\omega)$, and therefore the finestructure spectrum, is effectively zero for $\delta\omega \gtrsim 2\pi$, or $f \gtrsim 1/\delta$.

Sonic boom: $\delta = .001$ sec; $f \gtrsim 1000$ Hz

$\delta = .002$ sec; $f \gtrsim 500$ Hz

$\Rightarrow D(r,\omega) \simeq 0$

8. SUMMARY

The main results of this investigation can be summarized as follows. First, the approach described here has been found to yield fairly concise, general expressions for the correlation and spectral density functions of the finestructure. Second, the results for the case of a ramp-function incident wave agree with the observation that the important time scales associated with the finestructure of sonic-boom and other types of transient waves are generally comparable to, or greater than, the rise time. Finally, the results (again for the ramp-function incident wave) indicate that an increase in the rise time of the incident wave is associated with a reduction in the high-frequency content of the finestructure.

This last result, assuming that it holds in the three-dimensional case as well, has implications as regards the idea of shaping the sonic boom in order to reduce its annoyance. It suggests that increasing the rise time of the sonic boom will make it quieter, even in the presence of turbulence.